

Noether and Lie symmetries for charged perfect fluids

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Abstract. We study the underlying nonlinear partial differential equation that governs the behaviour of spherically symmetric charged fluids in general relativity. We investigate the conditions for the equation to admit a first integral or be reduced to quadratures using symmetry methods for differential equations. A general Noether first integral is found. We also undertake a comprehensive group analysis of the underlying equation using Lie point symmetries. The existence of a Lie symmetry is subject to solving an integro-differential equation in general; we investigate the conditions under which it can be reduced to quadratures. Earlier results for uncharged fluids and particular first integrals for charged matter are regained as special cases of our treatment.

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1. Introduction

The Einstein-Maxwell system of equations plays a central role in relativistic astrophysics when describing spherically symmetric gravitational fields in static manifolds. In these situations we are modelling charged compact objects with strong gravitational fields such as dense neutron stars. Some recent investigations, including the models of Ivanov [1], Sharma *et al* [2] and Thirukkanesh and Maharaj [3], indicate that the electromagnetic field significantly affects physical quantities in relativistic stellar systems: equations of state, redshifts, luminosities, stability and maximum masses of compact relativistic stars. The presence of electric charge is a necessary ingredient in the structure and gravitational evolution of stars composed of quark matter as illustrated in the treatments of Mak and Harko [4] and Komathiraj and Maharaj [5]. These applications emphasise the need for exact solutions of the Einstein-Maxwell system in modelling the formation and evolution of charged astrophysical objects. Lasky and Lun [6, 7], amongst others, studied the role of electromagnetic fields in gravitational collapse, formation of black holes and the existence of naked singularities. Electric fields cannot be ignored in spherical gravitational collapse with phases of intense dynamical activity and particle interaction as shown in the treatments of Di Prisco *et al* [8] and Herrera *et al* [9]. Maxwell's equations also play an important role in cosmological models in higher dimensions, brane world models and wormhole configurations. In this context see the treatment of De Felice and Ringeval [10].

When solving the Einstein field equations with neutral matter distributions, we often make the assumption that the spacetime is shear-free and spherically symmetric. Kustaanheimo and Qvist [11] were the first to present a general class of solutions. The generalisation to include the presence of the electromagnetic field may be easily achieved. The field equations are reducible to a single partial differential equation. A review of known charged solutions, with a Friedmann limit, is given by Krasinski [12]. Srivastava [13] and Sussmann [14, 15] undertook a detailed study of the mathematical and physical features of the Einstein-Maxwell system in spherical symmetry. Wafo Soh and Mahomed [16] used symmetry methods to systematically study the underlying partial differential equation. They showed that all previously known solutions can be related to a Noether point symmetry

The main aim of this paper is to study the integrability properties of the underlying partial differential equation for the Einstein-Maxwell system using symmetry methods. Both Noether and Lie point symmetries of the governing equation are considered. Noether symmetries have the interesting property of being associated with physically relevant conservation laws in a direct manner via the well-known Noether theorem. The Lie symmetries are more general, providing a larger set of symmetry generators in general, but do not guarantee integrability and reduction to quadrature in a straight forward manner. In section 2, we reduce the Einstein-Maxwell field equations to a single nonlinear second order partial differential equation that governs the behaviour of charged fluids. This is achieved by utilising the generalised transformation due to

Faulkes [17]. The resulting partial differential equation can be treated as an ordinary differential equation as in the case of uncharged fluids. In section 3, we present a first integral of the governing equation obtained earlier by generalising the technique of Srivastava [13] first used for uncharged fluids. This first integral is subject to two integrability conditions expressed as nonlinear integral equations. We transform the integral conditions into a fourth order differential equation. In section 4, we analyse the governing equation for Noether symmetries via its Lagrangian and this analysis yields a general Noether first integral for this equation. We then establish the relationship between the Noether first integral and the first integral (which had been obtained earlier using an *ad hoc* approach). In section 5, we undertake a comprehensive Lie symmetry analysis of the governing equation to investigate the conditions under which it can be reduced to quadratures. We show how the Noether results are a subset of the Lie analysis results. Lastly, in section 6, we discuss the results obtained and relate some invariant solutions to known results.

2. Field equations

We analyse the shear-free motion of a fluid distribution in the presence of an electric field. It is possible to choose coordinates $x^i = (t, r, \theta, \phi)$ such that the line element can be written in the form

$$ds^2 = -e^{2\nu(t,r)} dt^2 + e^{2\lambda(t,r)} \left[dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (1)$$

which is simultaneously comoving and isotropic. Since an electromagnetic field is present, the Einstein field equations are supplemented with Maxwell's equations to describe a self-gravitating charged fluid. The Einstein field equations for a charged perfect fluid can be written as the system

$$\rho = 3 \frac{\lambda_t^2}{e^{2\nu}} - \frac{1}{e^{2\lambda}} \left(2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r} \right) - \frac{E^2}{r^4 e^{4\lambda}} \quad (2a)$$

$$p = \frac{1}{e^{2\nu}} \left(-3\lambda_t^2 - 2\lambda_{tt} + 2\nu_t \lambda_t \right) + \frac{1}{e^{2\lambda}} \left(\lambda_r^2 + 2\nu_r \lambda_r + \frac{2\nu_r}{r} + \frac{2\lambda_r}{r} \right) + \frac{E^2}{r^4 e^{4\lambda}} \quad (2b)$$

$$p = \frac{1}{e^{2\nu}} \left(-3\lambda_t^2 - 2\lambda_{tt} + 2\nu_t \lambda_t \right) + \frac{1}{e^{2\lambda}} \left(\nu_{rr} + \nu_r^2 + \frac{\nu_r}{r} + \frac{\lambda_r}{r} + \lambda_{rr} \right) - \frac{E^2}{r^4 e^{4\lambda}} \quad (2c)$$

$$0 = \nu_r \lambda_t - (\lambda_t)_r \quad (2d)$$

Maxwell's equations imply that

$$E = r^2 e^{\lambda-\nu} \Phi_r, \quad E_r = \sigma r^2 e^{3\lambda} \quad (3)$$

In the above ρ is the energy density, p is the isotropic pressure, σ is the proper charge density and E is the total charge contained within the sphere of radius r centered around the origin. The matter variables are measured relative to the four-velocity $\mu^a = (e^{-\nu}, 0, 0, 0)$. Subscripts refer to partial derivatives. Observe that $\Phi_r = F_{10} (= -F_{01})$ is the only nonzero component of the Faraday electromagnetic field tensor $F_{ab} = \phi_{b;a} - \phi_{a;b}$ where we have chosen $\phi_a = (\Phi(t, r), 0, 0, 0)$ as the simplest

choice that allows for a nonvanishing electric field. The coupled equations (2a)-(2d), (3) comprise the Einstein-Maxwell system for the metric (1) in terms of the variables $\phi, p, E, \sigma, \nu, \lambda$.

The Einstein-Maxwell field equations can be reduced to a simpler system as shown by Srivastava [18]. After considerable simplification the Einstein field equations (2a)-(2d) can be written in the equivalent form

$$\rho = 3e^{2h} - e^{-2\lambda} \left(2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r} \right) - \frac{E^2}{r^4 e^{4\lambda}} \quad (4a)$$

$$p = \frac{1}{\lambda_t e^{3\lambda}} \left[e^\lambda \left(\lambda_r^2 + \frac{2\lambda_r}{r} \right) - e^{3\lambda+2h(t)} - \frac{E^2 e^{-\lambda}}{r^4} \right]_t \quad (4b)$$

$$e^\nu = \lambda_t e^{-h} \quad (4c)$$

$$e^\lambda \left(\lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r} \right) = -\tilde{F} - \frac{2E^2 e^{-\lambda}}{r^4} \quad (4d)$$

for a charged relativistic fluid, where $h = h(t)$ and $\tilde{F} = \tilde{F}(r)$ are arbitrary constants of integration. We have the condition $\lambda_t \neq 0$ from (4b) so that the model must be time dependent, and the spacetime cannot become static. Observe that (4d) is essentially the generalised condition of pressure isotropy which includes a term corresponding to the electric field. To find an exact solution to the system (4a)-(4d) we need to explicitly integrate (4d) to determine λ , and σ follows from (3).

To continue we need to express (4d) in a simpler form. We use a transformation first utilised by Faulkes [17] for neutral fluids. We let

$$x = r^2 \quad (5a)$$

$$y = e^{-\lambda} \quad (5b)$$

$$f(x) = \frac{\tilde{F}(r)}{4r^2} \quad (5c)$$

$$g(x) = \frac{E^2}{2r^6} \quad (5d)$$

With the help of (5a)-(5d) we can write (4d) in a more compact form

$$y'' = f(x)y^2 + g(x)y^3 \quad (6)$$

where primes denote differentiation with respect to the variable x . Equation (6) is the fundamental nonlinear partial differential equation which determines the behaviour of the self-gravitating charged fluid in general relativity. However, we can treat it as an ordinary differential equation as only derivatives with respect to x appear. If $g = 0$ then we regain a neutral fluid which has been studied by Maharaj *et al* [19] and Wafo Soh and Mahomed [20], amongst others.

3. A charged first integral

It is possible to find a first integral of (6) without choosing explicit forms of the functions $f(x)$ and $g(x)$. We use integration by parts, an approach adopted by Maharaj *et al* [19]

in investigating the integrability properties of the field equation $y'' = f(x)y^2$ of a neutral spherically symmetric shear-free fluid. We can integrate (6) by parts to obtain

$$y' = f_I y^2 + g_I y^3 - 2f_{II} y y' + 2f_{III} y'^2 + 2(f f_{II})_I y^3 + 2(g f_{II})_I y^4 - K_0 y^3 - K_1 y^4 + \tau_0(t) \quad (7)$$

For convenience we have used the notation

$$\int f(x) dx = f_I \quad \int g(x) dx = g_I \quad (8)$$

The result (7) is subject to the integrability conditions

$$K_0 = \frac{4}{3} f f_{III} + 2(f f_{II})_I + g_I \quad (9a)$$

$$K_1 = g f_{III} + 2(g f_{II})_I \quad (9b)$$

where K_0 and K_1 are constants and $\tau_0(t)$ is an arbitrary function of integration.

The integral equations (9a)-(9b) may be simplified. On setting

$$f_{III} = a \quad (10)$$

we can rewrite (9b) as

$$ga + 2(ga')_I = K_1$$

which has the solution

$$g = g_0 a^{-3} \quad (11)$$

where $a = a(x)$ and g_0 is an arbitrary constant. Equation (9a) now becomes

$$\frac{4}{3} a''' a + 2(a''' a')_I + (g_0 a^{-3})_I = K_0$$

or, if we want a purely differential equation,

$$aa^{(iv)} + \frac{5}{2} a' a''' = -\frac{3}{4} g_0 a^{-3} \quad (12)$$

We have therefore established that the integrability conditions (9a)-(9b) can be transformed into the fourth order ordinary differential equation (12). Using (9a)-(9b), (10) and (11) we rewrite the first integral (7) of (6) as

$$\tau_0(t) = y' - a'' y^2 + 2a' y y' - 2a y'^2 + \frac{4}{3} a a''' y^3 + g_0 a^{-2} y^4, \quad (13)$$

subject to condition (12). A solution of (12) will give a , and then f and g will be found from (10) and (11) respectively. A full analysis of this case can be found in Kweyama *et al* [21], and we will not repeat those results here.

4. Noether symmetries and integration

Given that Noether's theorem generates first integrals in a direct manner, we now investigate (6) for Noether symmetries in order to generate first integrals. Note that, if a second order ordinary differential equation

$$y'' = N(x, y, y') \quad (14)$$

has a Lagrangian $\mathcal{L}(x, y, y')$ then (14) is equivalent to the Euler-Lagrange equation

$$\frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial y'} \right) - \frac{\partial \mathcal{L}}{\partial y} = 0$$

The determining equation for a Noether point symmetry

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad (15)$$

corresponding to a Lagrangian $\mathcal{L}(x, y, y')$ of (14) is

$$X^{[1]} \mathcal{L} + \left(\frac{d\xi}{dx} \right) \mathcal{L} = \frac{dF}{dx}$$

where

$$X^{[1]} = X + \left(\frac{d\eta}{dx} - y' \frac{d\xi}{dx} \right) \frac{\partial}{\partial y'}$$

and $F = F(x, y)$ is a gauge function. It is also known that if there is a Noether symmetry corresponding to a Lagrangian of an equation (14), then (14) can be reduced to quadratures. This is a critical advantage of Noether point symmetries as emphasised by Wafo Soh and Mahomed [16]. The Noether first integral I_N associated with the Noether point symmetry (15) is given by

$$I_N = \xi(x, y) \mathcal{L} + (\eta(x, y) - y' \xi(x, y)) \mathcal{L}_{y'} - F \quad (16)$$

in terms of \mathcal{L} and F .

For equation (6) a Lagrangian is

$$\mathcal{L} = \frac{1}{2} y'^2 + \frac{1}{3} f(x) y^3 + \frac{1}{4} g(x) y^4 \quad (17)$$

The Lagrangian \mathcal{L} admits the Noether point symmetry

$$G = a \frac{\partial}{\partial x} + (by + c) \frac{\partial}{\partial y} \quad (18)$$

provided

$$b = \frac{1}{2} a' \quad (19a)$$

$$g = g_1 a^{-3} \quad (19b)$$

$$f = a^{-5/2} \left(f_1 - 3g_1 \int ca^{-3/2} dx \right) \quad (19c)$$

$$a''' = 4a^{-5/2} c \left(f_1 - 3g_1 \int ca^{-3/2} dx \right) \quad (19d)$$

$$c = C_0 + C_1 x \quad (19e)$$

$$F = \frac{1}{4} a'' y^2 + c' y \quad (19f)$$

where $a = a(x)$, $b = b(x)$, $c = c(x)$ and f_1 and g_1 are constants. From (19c) and (19d) we have

$$f = \frac{a'''}{4c}$$

When we differentiate (19d) once with respect to x we obtain

$$caa^{(iv)} + \frac{5}{2}ca'a''' - ac'a''' = -12g_1c^3a^{-3} \quad (20)$$

Applying the transformation

$$X = \frac{C_1}{C_0 + C_1x}, \quad A = \frac{aC_1^2}{(C_0 + C_1x)^2} \quad (21)$$

reduces (20) to

$$AA^{(iv)} + \frac{5}{2}A'A''' = -12g_1C_1^2A^{-3} \quad (22)$$

which is in the form of (12).

From (16) we have that (6) admits the Noether first integral

$$\begin{aligned} I_N = & a \left[\frac{1}{2}y'^2 + \frac{1}{3}y^3a^{-(5/2)} \left(f_1 - 3g_1 \int ca^{-(3/2)}dx \right) + \frac{1}{4}g_1a^{-3}y^4 \right] \\ & + \left(\frac{1}{2}a'y + c - ay' \right) y' - \frac{1}{4}a''y^2 - c'y \end{aligned} \quad (23)$$

Substituting (19d) in (23) yields

$$I_N = cy' - \frac{1}{4}a''y^2 + \frac{1}{2}a'yy' - \frac{1}{2}ay'^2 + \frac{aa'''}{12c}y^3 + \frac{1}{4}g_1a^{-2}y^4 - c'y. \quad (24)$$

On comparing our first integral (13) and the Noether first integral (24) we observe that, with $c = 1/4$, $f_0 = f_1$ and $g_0 = g_1$,

$$\tau_0(t) = 4I_N$$

As a result, our earlier *ad hoc* approach yielded a first integral which is a special case of that obtained via Noether's theorem. This further implies that (13) admits the Noether symmetry

$$Y = 4a \frac{\partial}{\partial x} + (2a'y + 1) \frac{\partial}{\partial y}$$

thus supporting the results of our approach, and the fact that (6) could be reduced to quadratures.

5. Lie analysis

As a final attempt at solving (6) we undertake a Lie symmetry analysis [22]-[26]. Lie symmetries are usually a larger set of symmetries for a problem as opposed to the set of Noether symmetries. However, the disadvantage is that no simple formula exists to find first integrals associated with Lie symmetries – direct integration of (often difficult) equations is usually needed. However, as we show here, this approach allows for a larger class of solutions than the Noether approach.

It is a simple matter to verify that (18) is a symmetry of (6) and the relationship among the functions $a(x), b(x), c(x), f(x)$ and $g(x)$ is given by the following system of ordinary differential equations

$$a'' = 2b' \quad (25a)$$

$$b'' = 2fc \quad (25b)$$

$$c'' = 0 \quad (25c)$$

$$af' + (2a' + b)f = -3cg \quad (25d)$$

$$ag' + (2a' + 2b)g = 0 \quad (25e)$$

From (25a)

$$2b = a' + \alpha$$

where α is an arbitrary constant, from (25a) and (25b)

$$f = \frac{a'''}{4c}, \quad (26)$$

from (25c) we have (19e) again, and finally from (25e)

$$g = g_2 a^{-3} \exp\left(-\int \frac{\alpha dx}{a}\right) \quad (27)$$

where g_2 is an arbitrary constant. By integrating (25d) and using (27) we obtain an alternative form of (26), and it is

$$f = a^{-5/2} \exp\left(-\int \frac{\alpha dx}{2a}\right) \left[f_2 - 3g_2 \int ca^{-(3/2)} \exp\left(\int \frac{\alpha dx}{a}\right) dx \right] \quad (28)$$

where f_2 is an arbitrary constant of integration. From (25d), (26) and (27) we observe that a is a solution of the equation

$$caa^{(iv)} + \left[c \left(\frac{5a'}{2} + \frac{\alpha}{2} \right) - c'a \right] a''' = -12g_2 c^3 a^{-3} \exp\left(-\int \frac{\alpha dx}{a}\right) \quad (29)$$

Alternatively (29) may be obtained by equating equations (26) and (28) and differentiating once with respect to x .

Observe the similarity between the results obtained here in the case of Lie analysis and those obtained in the previous section on Noether symmetries. The main difference is the occurrence of the parameter α in the Lie analysis. Thus, we have a more general result (as expected) in the Lie analysis as compared to the Noether analysis. In addition, this result is even more general than that obtained in Kweyama *et al* [21] as our $c(x)$ is nonconstant.

The transformation which converts the symmetry (18) to $\frac{\partial}{\partial X}$ makes (6) autonomous. The simplest expression of this transformation is

$$X = \int \frac{dx}{a}, \quad Y = y \exp\left(-\int \frac{b dx}{a}\right) - \int \frac{c}{a} \exp\left(-\int \frac{b dx}{a}\right) dx \quad (30)$$

Using (30) equation (6) becomes

$$Y'' + \alpha Y' + \left(M + \frac{\alpha^2}{4} \right) Y = f_2 Y^2 + g_2 Y^3 + N \quad (31)$$

where f_2 and g_2 are arbitrary constants introduced in (27) and (28) respectively. The quantities M and N are arbitrary constants that arise in integrations of (29) and are given by

$$M = \frac{1}{2}aa'' - \frac{1}{4}a'^2 - 2f_2I + 3g_2I^2 \quad (32)$$

and

$$\begin{aligned} N = & -a^{-1/2} \exp\left(-\int \frac{\alpha dx}{2a}\right) \left(ac' - \frac{1}{2}a'c + \frac{1}{2}\alpha c\right) \\ & - \left(\frac{1}{2}aa'' - \frac{1}{4}a'^2 + \frac{\alpha^2}{4}\right) I + f_2I^2 - 2g_2I^3 \end{aligned} \quad (33)$$

where

$$I = \int ca^{-(3/2)} \exp\left(-\int \frac{\alpha dx}{2a}\right) dx \quad (34)$$

Note that in the neutral perfect fluid case we must have $g_2 = N = 0$ and (31) reduces to that of Maharaj *et al* [19]. However, it is difficult to make direct comparison to the results therein as the full equations are not always given, and the equation referencing is not always clear.

To proceed further, we need to analyse equation (29). We note that the form of (29) was obtained under the assumption that $c \neq 0$. We consider both nonzero c and vanishing c in turn in our subsequent analysis.

5.1. Case I: $c \neq 0$

When $C_0 \neq 0$ and $C_1 = 0$, (29) may be written as

$$aa^{(iv)} + \frac{1}{2}(5a' + \alpha)a''' = -12g_2a^{-3} \exp\left(-\int \frac{\alpha dx}{a}\right) \quad (35)$$

Rescaling a and x in (35) yields

$$aa^{(iv)} + \frac{1}{2}(5a' + \alpha)a''' = a^{-3} \exp\left(-\int \frac{\alpha dx}{a}\right) \quad (36)$$

when $\alpha \neq 0$ and

$$aa^{(iv)} + \frac{5}{2}a'a''' = a^{-3} \quad (37)$$

when $\alpha = 0$.

When $C_0 \neq 0$ and $C_1 \neq 0$, we apply transformation (21) to (29) and we obtain (again with the rescaling of A and X)

$$AA^{(iv)} + \frac{1}{2}(5A' - \alpha)A''' = A^{-3} \exp\left(\int \frac{\alpha dX}{A}\right) \quad (38)$$

when $\alpha \neq 0$ and

$$AA^{(iv)} + \frac{5}{2}A'A''' = A^{-3}$$

when $\alpha = 0$. Changing the sign of α in (38) brings it to the form of (36), and so the critical equations are (36) and (37).

5.1.1. *Case Ia:* $\alpha = 0$. If $\alpha = 0$ then (31) becomes

$$Y'' + MY = f_2 Y^2 + g_2 Y^3 + N \quad (39)$$

The solution of (39) is expressed as the quadrature

$$X - X_0 = \int \frac{dY}{\sqrt{\frac{1}{2}g_2 Y^4 + \frac{2}{3}f_2 Y^3 - MY^2 + 2NY + L}} \quad (40)$$

where L is an arbitrary constant introduced in the first integration of (39). A full discussion of the evaluation of this quadrature can be found in Kweyama *et al* [21]. If we invoke the same transformations in obtaining (39) to the first integral (24) we obtain

$$-L = \frac{1}{4}g_1 Y^4 + \frac{1}{3}f_1 Y^3 - \frac{1}{2}MY^2 + NY - \frac{1}{2}Y'^2$$

which is the intermediate integration between (39) and (40). This again indicates that the Noether results are a subset of the Lie results.

We now have to determine $a(x)$. We make the observation that if $\alpha = 0$ and $g_0 = 16g_2$ and $g_1 = g_2/C_1^2$, (12) and (22) become (35) which reduces to (37). We therefore consider (37) for further analysis and reduction to quadrature. In carrying out the Lie analysis of (37), using PROGRAM LIE [27], we find that it has two Lie point symmetries, namely

$$G_1 = \frac{\partial}{\partial x}$$

$$G_2 = x \frac{\partial}{\partial x} + \frac{4}{5}a \frac{\partial}{\partial a}$$

Usually, when an n th order equation admits an $m < n$ dimensional Lie algebra of symmetries, there is little hope for the solution of the equation via those symmetries. However, in this case we are able to reduce the equation due to the presence of hidden symmetries [23].

The symmetry G_1 determines the variables for reduction

$$u = a, \quad v = a'$$

and the reduced equation is

$$u^4 v^3 v''' + 4u^4 v^2 v' v'' + \frac{5}{2}u^3 v^3 v'' + u^4 v v'^3 + \frac{5}{2}u^3 v^2 v'^2 - 1 = 0$$

This equation admits the following two symmetries

$$U_1 = u \frac{\partial}{\partial u} - \frac{1}{4}v \frac{\partial}{\partial v}$$

$$U_2 = 2u^2 \frac{\partial}{\partial u} + uv \frac{\partial}{\partial v}$$

The variables for reduction via U_2 are

$$r = u^{-\frac{1}{2}}v \quad s = u^{\frac{3}{2}}v' - \frac{1}{2}u^{\frac{1}{2}}v$$

and the reduced equation is

$$r^3 s^2 s'' + r^3 s s'^2 + 4r^2 s^2 s' + r s^3 - 1 = 0 \quad (41)$$

The Lie symmetry analysis of (41) yields the following two symmetries

$$\begin{aligned} X_1 &= r \frac{\partial}{\partial r} - \frac{s}{3} \frac{\partial}{\partial s} \\ X_2 &= \frac{\partial}{\partial r} - \frac{s}{r} \frac{\partial}{\partial s} \end{aligned}$$

The reduction variables generated by X_2 are

$$p = rs \quad q = rs' + s$$

and the reduced equation is

$$p^2 qq' + pq^2 - 1 = 0$$

with solution

$$q^2 = \frac{2}{p} + \frac{q_0}{p^2}$$

where q_0 is a constant. We can now invert these transformations to find the solution of (37). Alternatively, we can integrate (37) directly and write down the solution as

$$\begin{aligned} u_x &= a^{-3/2} = [G'(u)]^{-1} \\ x - x_0 &= G(u) \end{aligned}$$

where we have set

$$G(u) = \int \frac{du}{(\mathcal{K}_4 + \mathcal{K}_3 u + \mathcal{K}_2 u^2 - (1/6)\mathcal{K}_1 u^3 + (1/32)\mathcal{K}_0 u^4)^{3/2}}$$

and the $\mathcal{K}_i, i = 0, \dots, 3$ are constants of integration related to M, N, f_2 and g_2 and \mathcal{K}_4 is arbitrary. As pointed out earlier, this result was obtained in Kweyama *et al* [21], but for constant $c(x)$. In the case of nonconstant $c(x)$, the solution is the same, except that we replace a and x in this solution with A and X respectively. To obtain the solution to (29) (with $\alpha = 0$) we need to apply the inverse of (21).

5.1.2. Case Ib: $\alpha \neq 0$. When $\alpha \neq 0$, we cannot directly reduce (31) to quadratures. We need to investigate the constraints under which it possesses a second point symmetry. We find that if $f_2 \neq 0, g_2 \neq 0$ then (31) has the following two symmetries

$$G_1 = \frac{\partial}{\partial X} \tag{42a}$$

$$G_2 = e^{(\alpha/3)X} \frac{\partial}{\partial X} - e^{(\alpha/3)X} \left(\frac{\alpha}{3} Y + \frac{\alpha f_2}{9g_2} \right) \frac{\partial}{\partial Y} \tag{42b}$$

provided the following conditions are satisfied

$$M = -\frac{f_2^2}{3g_2} - \frac{\alpha^2}{36}, \quad N = \frac{f_2^3}{27g_2^2} - \frac{2\alpha^2 f_2}{27g_2} \tag{43}$$

Utilising (42b) we obtain the transformation

$$\mathcal{X} = -\frac{3}{\alpha} e^{-(\alpha/3)X}, \quad \mathcal{Y} = e^{(\alpha/3)X} \left(Y + \frac{f_2}{3g_2} \right)$$

and equation (31) becomes

$$\mathcal{Y}'' = g_2 \mathcal{Y}^3$$

with solution

$$\mathcal{X} - \mathcal{X}_0 = \int \frac{d\mathcal{Y}}{\sqrt{\frac{g_2}{2}\mathcal{Y}^4 + \mathcal{C}}}$$

The values in (43) look rather peculiar. However, it is interesting to note that these values correspond directly to a simplification of the eigenvalue problem associated with a dynamical systems analysis of (31)!

When $c \neq 0$, $g_2 = 0$, $f_2 \neq 0$ we find that $g \equiv 0$ (and so we are in the neutral perfect fluid realm). Now (31) has the following two symmetries

$$Y_1 = \frac{\partial}{\partial X} \tag{44a}$$

$$Y_2 = e^{(\alpha/5)X} \frac{\partial}{\partial X} + \left(\frac{\alpha^3}{500f_2} + \frac{\alpha M}{5f_2} - \frac{2\alpha}{5} Y \right) e^{(\alpha/5)X} \frac{\partial}{\partial Y} \tag{44b}$$

provided the following condition is satisfied

$$\begin{aligned} N &= \frac{M^2}{4f_2} + \frac{\alpha^2 M}{8f_2} + \frac{49\alpha^4}{40000f_2} \\ &= \frac{1}{4f_2} \left(M + \frac{\alpha^2}{4} \right)^2 - \frac{36\alpha^4}{2500f_2} \end{aligned} \tag{45}$$

This condition is equivalent to the one obtained by Mellin *et al* [24] for the case where $n = 2$ in their analysis of the generalised Emden-Fowler equation. We use (44b) to obtain the following transformation

$$\mathcal{X} = -\frac{5}{\alpha} e^{(-\alpha/5)X}, \quad \mathcal{Y} = \left(Y - \frac{M}{2f_2} - \frac{\alpha^2}{200f_2} \right) e^{(2\alpha/5)X}$$

which, together with (45) reduces equation (31) to

$$\mathcal{Y}'' = f_2 \mathcal{Y}^2$$

with solution

$$\mathcal{X} - \mathcal{X}_0 = \int \frac{d\mathcal{Y}}{\sqrt{\frac{2f_2}{3}\mathcal{Y}^3 + \mathcal{C}_1}}$$

We take this opportunity to make two minor corrections to the work of Mellin *et al* [24]: While the expression for N given by their equation (7.11) is correct, it is obtained by multiplying their equation (7.9) by $a \int da^{-3/2} \exp \left[\frac{1}{2} \int (p - 2C_0/a) dx \right] dx$ and then integrating, not multiplying by $a \int da^{-3/2} dx$ as indicated in their paper. Also, the coefficient of C_0 should be 2 in their equation (7.8).

5.2. Case II: $c = 0$

From (25a), (25b), (25d) and (25e) we have

$$b = \frac{1}{2}(a' + \alpha) \quad (46a)$$

$$a = a_0 + a_1x + a_2x^2 \quad (46b)$$

$$f = f_2a^{-5/2} \exp\left(-\int \frac{\alpha dx}{2a}\right) \quad (46c)$$

$$g = g_2a^{-3} \exp\left(-\int \frac{\alpha dx}{a}\right) \quad (46d)$$

The symmetry (18) now takes the form

$$G = a \frac{\partial}{\partial x} + \frac{1}{2}(a' + \alpha)y \frac{\partial}{\partial y}$$

Using the transformation

$$X = \int \frac{dx}{a}, \quad Y = ya^{-1/2} \exp\left(-\int \frac{\alpha dx}{2a}\right)$$

equation (6) is transformed into the autonomous equation

$$Y'' + \alpha Y' + \beta Y = f_2 Y^2 + g_2 Y^3 \quad (47)$$

where

$$\beta = \frac{1}{4}(\alpha^2 - \Delta), \quad \Delta = a_1^2 - 4a_0a_2$$

In carrying out the standard Lie point symmetry analysis on (47) we have the following cases:

5.2.1. Case IIa. If $f_2 \neq 0$, $g_2 \neq 0$, (47) has the following two symmetries

$$G_1 = \frac{\partial}{\partial X} \quad (48a)$$

$$G_2 = e^{(\alpha/3)X} \frac{\partial}{\partial X} - e^{(\alpha/3)X} \left(\frac{\alpha}{3}Y + \frac{2\alpha^3}{9f_2} \right) \frac{\partial}{\partial Y} \quad (48b)$$

provided the following conditions apply

$$\beta = -\frac{4\alpha^2}{9}, \quad g_2 = \frac{f_2^2}{2\alpha^2} \quad (49)$$

We use (48b) to obtain the following transformation

$$\mathcal{X} = -\frac{3}{\alpha}e^{-(\alpha/3)X}, \quad \mathcal{Y} = e^{-(\alpha/3)X} \left(Y + \frac{2\alpha^2}{3f_2} \right) \quad (50)$$

Using (49) and (50) the equation (47) becomes

$$\mathcal{Y}'' = \frac{f_2^2}{2\alpha^2} \mathcal{Y}^3 \quad (51)$$

and the solution of (51) is

$$\mathcal{X} - \mathcal{X}_0 = \int \frac{d\mathcal{Y}}{\sqrt{\frac{f_2^2}{4\alpha^2} \mathcal{Y}^4 + \mathcal{C}}}$$

5.2.2. *Case IIb.* If $f_2 = 0$, $g_2 \neq 0$ (which implies that $f = 0$), then (47) has the following two symmetries

$$G_1 = \frac{\partial}{\partial X} \quad (52a)$$

$$G_2 = e^{(\alpha/3)X} \frac{\partial}{\partial X} - \frac{\alpha}{3} e^{(\alpha/3)X} Y \frac{\partial}{\partial Y} \quad (52b)$$

subject to the following condition

$$\beta = \frac{2\alpha^2}{9} \quad (53)$$

Using (52b) we obtain the following transformation

$$\mathcal{X} = -\frac{3}{\alpha} e^{-(\alpha/3)X}, \quad \mathcal{Y} = e^{(\alpha/3)X} Y \quad (54)$$

Using (53) and (54) equation (47) is transformed to

$$\mathcal{Y}'' = g_2 \mathcal{Y}^3 \quad (55)$$

and the solution of (55) is

$$\mathcal{X} - \mathcal{X}_0 = \int \frac{d\mathcal{Y}}{\sqrt{\frac{g_2}{2} \mathcal{Y}^4 + \mathcal{C}}}$$

This is an intrinsically charged result - there is no uncharged analogue.

5.2.3. *Case IIc.* If $f_2 \neq 0$, $g_2 = 0$ (which implies $g = 0$), then (47) has two symmetries provided

$$\beta = \pm \frac{6\alpha^2}{25} \quad (56)$$

and can be transformed to

$$\mathcal{Y}'' = f_2 \mathcal{Y}^2$$

which has the solution

$$\mathcal{X} - \mathcal{X}_0 = \int \frac{d\mathcal{Y}}{\sqrt{\frac{2f_2}{3} \mathcal{Y}^3 + \mathcal{C}}}$$

This result was previously obtained in the neutral case by Maharaj *et al* [19]. A consequence of (56) is that $\Delta > 0$ (Note that this is not imposed on (47) as was done in Maharaj *et al* [19].) and hence a in (46b) has real roots.

6. Discussion

We have undertaken a comprehensive analysis of (6) in order to determine which forms of the functions f and g would lead to first integrals and/or solutions of the equation. We reviewed our previous *ad hoc* approach, and showed that those results were contained in the results obtained via Noether's theorem. This occurred when the function c obtained in the Noether analysis was set to $\frac{1}{4}$. These latter results were then shown to be further

contained with those obtained via the Lie analysis. This occurs when $\alpha = 0$. In this case (27) and (28) take the forms of (19b) and (19c) respectively, and so (19b) and (19c) are special cases of (27) and (28) respectively.

Usefully, the first two approaches, namely the *ad hoc* approach and the Noether symmetry analysis, yielded first integrals directly. The final (Lie) approach needed two stages of analysis in order to reduce the equation to quadratures.

We were able to completely analyse (6) for Lie point symmetries in an exhaustive analysis. All possible cases for the functions f and g were analysed. We showed that these results reduced to the neutral case results ($g = 0$) of Maharaj *et al* [19] in most cases. However, we were also able to find an inherently charged case in §5.2.2 that has no uncharged analogue.

While a complete analysis was produced for the case of $\alpha = 0$, only a partial analysis could be performed when $\alpha \neq 0$. Nonetheless, we were still able to provide conditions under which (6) could be reduced to quadratures. It still remains to solve (36). The main difficulty is that this equation is an integro-differential equation and such equations are notoriously difficult to solve. However, we can still reduce (36) to a first order equation by eliminating a'' from (32) and (33). Further work in this direction is ongoing.

Note that, in the case that $c = 0$, we were able to find constraints under which we could reduce (6) to quadratures with f and g given explicitly and $\alpha \neq 0$. It is interesting to observe that the constraints we found in all subcases of §5.2 forced the quadratic a in (46b) to have real roots.

To complete our analysis, we take the symmetries calculated in §5.1.1 and investigate the possibility of group invariant solutions [25, 26] of (37). The only two results of significance are that

$$x - x_0 = \pm \int \frac{da}{\left[-\left(\frac{32}{3}\right)^{1/3} a^{-(1/3)} + K_1 a^{2/3} \right]^{3/4}} \quad (57)$$

and

$$a = -\left(\frac{625}{24}\right)^{1/5} x^{4/5} \quad (58)$$

are invariant solutions of (37). The first solution can only be given implicitly and so is not of much practical use. However, if we let $c = \frac{1}{4}$ in (26), $\alpha = 0$ in (27) and $K_1 = 0$ in (57), then we obtain explicit forms of f and g . Also if we substitute (58) into (26) and (27) we can explicitly determine f and g . In both cases we obtain the forms

$$f(x) \propto (x - x_0)^{-(11/5)}, \quad g(x) \propto (x - x_0)^{-(12/5)} \quad (59)$$

These forms for f and g were earlier obtained by Kweyama *et al* [21] using an *ad hoc* approach which yielded a new charged first integral to the Einstein-Maxwell field equations. In this case, (6) admits two Lie point symmetries and can be reduced to quadratures.

We can also reuse (58) by invoking (21) to obtain

$$a = \kappa \left(\frac{c'}{c} \right)^{-(6/5)} \quad (60)$$

which is a solution to (29) provided

$$g_2 = -\frac{24\kappa^5}{625C_1^2}. \quad (61)$$

This corresponds to the choice

$$f(x) \propto (x - x_0)^{-(14/5)}, \quad g(x) \propto (x - x_0)^{-(18/5)} \quad (62)$$

for which (6) again has two Lie point symmetries and can be reduced to quadratures.

In summary, we have given a complete Noether and Lie point symmetry analysis of (6). For the Lagrangian (17) we were able to provide the most general Noether point symmetry, the most general first integral associated with this symmetry and indicated that this integral was equivalent to that found via an *ad hoc* approach, *i.e* (13). Finally we determined the most general Lie point symmetry admitted by (6) and gave conditions under which the equation could be reduced either to a first order equation, or to quadratures.

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